A FULLY DISCRETE BEM-FEM FOR THE EXTERIOR STOKES PROBLEM IN THE PLANE

SALIM MEDDAHI† AND FRANCISCO-JAVIER SAYAS‡

Abstract. We reformulate the Johnson–Nedelec approach for the exterior two-dimensional Stokes problem taking advantage of the parameterization of the artificial boundary. The main aim of this paper is the presentation and analysis of a fully discrete numerical method for this problem. This one responds to the needs of having efficient approximate quadratures for the weakly singular boundary integrals. We give a complete error analysis of both the Galerkin and fully discrete Galerkin methods.

Key words. exterior Stokes problem, mixed finite elements, boundary elements

AMS subject classifications. 65N30, 65F10

1. Introduction. The purpose of this paper is to introduce and analyze a new quadrature method for computing the Galerkin stiffness matrices arising from the BEM-FEM discretization of the exterior Stokes problem in the plane. The first BEM-FEM procedure for this problem was introduced by Sequeira in [13]. The formulation of Sequeira relays on the so-called one boundary integral approach introduced by Johnson and Nedelec for the Laplace equation [9].

The general Johnson–Nedelec procedure consists of dividing the unbounded domain into two subregions, a bounded inner region and an unbounded outer one, by introducing an auxiliary common boundary. This division is done so that the support of the right-hand side of the equation (i.e., the external forces) falls into the inner domain. An adapted Green formula, which makes use of the fundamental solution of the Stokes problem, gives an integral representation of the solution in the exterior domain. Next, this representation is used to deduce a nonlocal condition on the auxiliary boundary for the problem in the inner region. We point out that it is important to choose a smooth artificial interface in order to assure the compactness of the double-layer potential which is essential for the analysis of the discrete problem.

From the point of view of implementation, one of the most time-consuming tasks is the matrix assembly process which requires the computation of integrals with nearly singular and smooth integrands over the auxiliary boundary. The design of efficient algorithms for this task is of great importance in order to improve the practicability of the method. This paper has the overall aim of providing a fully discrete Galerkin method requiring few kernel evaluations while preserving the stability and convergence properties which are obtained when the integrals are computed exactly.

The principal idea of our method consists of avoiding the consistency error due to the substitution of the auxiliary boundary by polygonal approximations when defining the discrete problem; see [9], [13]. To this end, we modify the variational formulation...
at the continuous level (as in [11]) by changing all terms on the artificial boundary to periodic functions by means of a smooth parametrization of this boundary. This equivalent reformulation of the continuous problem is especially important since it leads to a novel discrete Galerkin scheme which allows one to take advantage of techniques from [8], [5] to compute in the global matrix the coefficients corresponding to the boundary integrals by elemental quadrature formulas; cf. problem (9). Our discretization method relays on exact triangulations of the domain. Hence, curved triangles are needed all along the auxiliary interface. This is one of the principal difficulties we had to answer since, to the authors’ knowledge, there is no curved stable finite elements for the Stokes problem in the literature; see [7], [3]. Thus, we generalize the mixed element introduced in [2] for our needs.

The paper is organized as follows. In a first part, which consists of sections 2 to 4, we introduce the model problem and its Galerkin discretization and arrive directly to the fully discrete scheme. We intend by this to let the reader see, as quickly as possible, what quadrature rules are used in avoiding the technical aspects. The second part is devoted to the analysis of the Galerkin scheme (section 5) and the completely discrete problem (section 6).

Convention. In what follows small boldface letters (capital boldface, resp.) will denote vectors or vector-valued functions (matrices or matrix-valued functions, resp.). Vectors in $\mathbb{R}^2$ are always to be understood as column vectors, and subscripts will index their different components. The superscript $\top$ will denote transposition of a matrix. A dot will denote the Euclidean inner product in $\mathbb{R}^2$ and a colon the Euclidean inner product in the space $\mathbb{R}^{2 \times 2}$ of the real $2 \times 2$ matrices, i.e.,

$$u \cdot v := u_1 \top v = \sum_{i=1}^{2} u_i v_i, \quad A : B := \sum_{i,j=1}^{2} A_{i,j} B_{i,j}.$$ 

Throughout this paper $C$, with or without subscripts, denotes a generic constant independent of the discretization parameter $h$.

Sobolev spaces. In this work we will use the Hilbertian Sobolev spaces $H^m(\Omega)$ for $\Omega$ open set in the plane, endowed with their usual norms denoted by $\| \cdot \|_m,\Omega$. In the particular case $H^0(\Omega) = L^2(\Omega)$ we will denote by $(\cdot,\cdot)_{0,\Omega}$ its inner product. The spaces $W^{m,\infty}(\Omega)$ are those Sobolev spaces derived from $L^\infty(\Omega)$ (cf. [4]), their norms being denoted as $\| \cdot \|_{m,\infty,\Omega}$. On the other hand, we consider the periodic Sobolev spaces. Let $C^\infty$ be the space of 1-periodic infinitely often differentiable real-valued functions of a single variable. Given $g \in C^\infty$, we define its Fourier coefficients

$$\hat{g}(k) := \int_0^1 g(s) e^{-2k\pi i s} ds.$$ 

Then for $r \in \mathbb{R}$ we define the Sobolev space $H^r$ to be the completion of $C^\infty$ with the norm

$$\|g\|_r := \left( \sum_{k \in \mathbb{Z}} (1 + |k|^2)^r |\hat{g}(k)|^2 \right)^{1/2}.$$ 

Then it is well known (see [10]) that $H^r$ are Hilbert spaces and that $H^{r_1} \subset H^{r_2}$ if $r_1 > r_2$, the inclusion being dense and compact. Moreover the $H^0$-inner product

$$(\lambda, \mu) := \int_0^1 \lambda(s) \mu(s) ds$$

where $\lambda, \mu \in L^2(\Omega)$. That is, if $\lambda, \mu \in L^2(\Omega)$ then $\lambda \cdot \mu = \int_\Omega \lambda \mu ds$. This is one of the main points in the construction of variational formulations of the Dirichlet problem for the Laplacian and the biharmonic equation. The extension to Sobolev spaces is straightforward. In the particular case $H^0$-inner product

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can be extended to represent the duality of $H^{-r}$ and $H^r$ for all $r$. We will keep the same notation for this duality bracket.

Since we will be dealing with vector unknowns, we need product forms of some spaces. If $H$ is any of the previous function spaces, we will denote $H := H \times H$ endowed with the product norm and corresponding inner product (when this exists). We will use the same notation for the inner product and norm, since it will be clear from the context and notations used for functions, when scalar or vector functions are used.

2. **Statement of the problem.** Given two functions $p$ and $u$ we denote

\[
\nabla p := \begin{pmatrix} \frac{\partial p}{\partial x_1} \\ \frac{\partial p}{\partial x_2} \end{pmatrix}, \quad \nabla \cdot u := \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}, \quad \Delta u := \begin{pmatrix} \Delta u_1 \\ \Delta u_2 \end{pmatrix},
\]

where $\Delta := \nabla \cdot \nabla$. We also consider the Jacobian matrix and the displacement tensor (symmetrized Jacobian matrix)

\[Du := \left( \frac{\partial u_j}{\partial x_i} \right)_{1 \leq i,j \leq 2}, \quad E[u] := \frac{1}{2} (Du + Du^\top).\]

We emphasize that the operator $D$ is the transposed of the more commonly used form for the differential matrix. We make this choice in order to simplify some forthcoming calculations.

Let $\Omega$ be a bounded domain in $\mathbb{R}^2$ with Lipschitz boundary $\Gamma$ and let $\Omega'$ be its exterior, i.e., the complement of its closure in $\mathbb{R}^2$. The steady-state Stokes problem consists in finding a velocity field $u$ and a pressure field $p$, defined on $\Omega'$, satisfying

\[
\begin{align*}
-\Delta u + \nabla p &= f, \quad \text{in } \Omega', \\
\nabla \cdot u &= 0, \quad \text{in } \Omega', \\
\mathbf{u} &= 0, \quad \text{on } \Gamma, \\
\mathbf{u}(x) &= \mathcal{O}(1), \quad \text{as } |x| \to \infty.
\end{align*}
\]

(1)

We assume that the support of the external force function $f$ is bounded. We have also assumed that the dynamic viscosity equals 1.

As usual in boundary-field formulations we introduce an artificial boundary. Let $\Omega_0$ be a simply connected bounded domain containing both $\text{supp } f$ and $\overline{\Omega}$ and such that its boundary $\Gamma_0$ can be parameterized by a 1-periodic smooth function, namely, $x : \mathbb{R} \to \Gamma_0 \subset \mathbb{R}^2$ satisfying

\[|x'(s)| > 0 \quad \forall s \in \mathbb{R}, \quad x(t) \neq x(s), \quad 0 < |t - s| < 1.\]

Then, this parameterization allows us to define the inner parameterized trace onto $\Gamma_0$ as the unique extension of

\[
\gamma : C^\infty(\overline{\Omega}) \to H^0, \\
u \mapsto u|_{\Gamma_0}(x(\cdot))
\]

to the whole of $H^1(\Omega^-)$, where $\Omega^- := \Omega' \cap \Omega_0$. By Theorem 8.15 of [10] we have that $\gamma : H^1(\Omega^-) \to H^{1/2}$ is bounded and onto.
We introduce the function spaces
\[
H^{-\frac{1}{2}}_0 := \{ \mu \in H^{-\frac{1}{2}} : (\mu, c) = 0 \quad \forall c \in \mathbb{R}^2 \}, \\
H^1_0(\Omega^-) := \{ v \in H^1(\Omega^-) : v|_{\Gamma} = 0 \}, \\
L^2_0(\Omega^-) := \{ q \in L^2(\Omega^-) : (1, q)_0,\Omega^- = 0 \}
\]
and the bilinear form
\[
a(u, v) := 2 \int_{\Omega^-} E[u] : E[v].
\]

Consider the integral operators
\[
V_{g} := \int_0^1 V(\cdot, t)g(t)dt, \quad K_{g} := \int_0^1 K(\cdot, t)g(t)dt,
\]
where
\[
V(s, t) := \frac{1}{4\pi} \log |x(s) - x(t)|I + \frac{1}{4\pi |x(s) - x(t)|^2}(x(s) - x(t))(x(s) - x(t))^\top,
\]
\[
I \quad \text{being the 2 \times 2 identity matrix, and}
\]
\[
K(s, t) := \frac{|x'(t)|}{\pi} (x(s) - x(t)) \cdot n(t) \quad (x(s) - x(t))(x(s) - x(t))^\top.
\]

Notice that $K$ is $C^\infty$ and 1-periodic in both variables. These operators are parameterized versions of classical boundary integral operators for the Stokes problem (cf. [6]) and satisfy the following well-known properties.

**Lemma 1.** For all $\theta_1, \theta_2$, $K : H^{\theta_1} \to H^{\theta_2}$ is compact. For all $\theta$, $V : H^{\theta} \to H^{\theta+1}$ is bounded. Moreover, there exists $\alpha > 0$ such that
\[
(V\mu, \mu) \geq \alpha \|\mu\|_{\frac{1}{2}, \frac{1}{2}}, \quad \forall \mu \in H^{-\frac{1}{2}}_0.
\]

These are therefore the elements to introduce the problem this paper concerns itself with:

\[
\begin{align*}
\text{find } (u, p, \lambda) \in H^1_0(\Omega^-) \times L^2_0(\Omega^-) \times H^{\frac{1}{2}}_0, \quad &\text{s.t.} \\
\quad a(u, v) - (p, \nabla \cdot v)_{0,\Omega^-} - (\gamma v, \lambda) = (f, v)_{0,\Omega^-} \quad &\forall v \in H^1_0(\Omega^-), \\
(2V\lambda, \mu) + (\gamma u, \mu) - (2K\gamma u, \mu) = 0 \quad &\forall \mu \in H^{-\frac{1}{2}}_0, \\
(q, \nabla \cdot u)_{0,\Omega^-} = 0 \quad &\forall q \in L^2_0(\Omega^-).
\end{align*}
\]

This problem arises from parameterization of the equivalent problem in [13]. Once (2) is solved, the weak solution to (1) is given by $(u, p)$ in the interior domain and by an integral representation using $\lambda$ and $\gamma u$ in the exterior, i.e., outside $\Gamma_0$ (see [13]). By a simple variation of an argument in [13], it can be easily proven that problem (2) has a unique solution. Notice that $\lambda$ is a parameterized version of the outer normal stress on $\Gamma_0$ and that the asymptotic behavior at infinity in (1) is ensured by the zero mean value condition on $\lambda$.

Consider the product space $\mathcal{M} := H^1_0(\Omega^-) \times H^{-1/2}_0$ endowed with the usual product Hilbert topology. Let $A : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ be the bounded bilinear form
\[
A(\pi, \eta) := a(u, v) - (\lambda, \gamma v) + (2V\lambda, \mu) + (\gamma u, \mu) - (2K\gamma u, \mu),
\]
where we have denoted \( \overline{\mathbf{u}} := (\mathbf{u}, \lambda) \), \( \overline{\mathbf{v}} := (\mathbf{v}, \mu) \) as we will do in what follows. We consider also \( L : \mathcal{M} \to \mathbb{R} \) and \( D : \mathcal{M} \times L^2_0(\Omega^-) \to \mathbb{R} \) given by

\[
L(\overline{\mathbf{v}}) := (\mathbf{f}, \mathbf{v})_{0, \Omega^-}, \quad D(\overline{\mathbf{v}}, q) := (\nabla \cdot \mathbf{u}, q)_{0, \Omega^-}.
\]

Then (2) is equivalent to

\[
\begin{align*}
\text{find} \ (\overline{\mathbf{u}}, p) \in \mathcal{M} \times L^2_0(\Omega^-) & \ 	ext{s.t.} \\
A(\overline{\mathbf{u}}, \overline{\mathbf{v}}) - D(\overline{\mathbf{v}}, p) &= L(\overline{\mathbf{v}}) \\
D(\overline{\mathbf{u}}, q) &= 0
\end{align*}
\]

(3)

3. A BEM-FEM discretization.

3.1. Curved triangulation of the bounded domain. For simplicity of exposition, we restrict ourselves to polygonal boundaries \( \Gamma \). In what follows, we construct a triangulation \( \tau_h \) covering exactly the domain \( \Omega^- \). Given \( h := 1/N, \) with \( N \) a positive integer, let \( t_1 := i h \) be the induced uniform partition of \( \mathbb{R} \). We denote by \( \Omega_h \) the polygonal domain whose vertices lying on \( \Gamma_0 \) are \( \Delta_h := \{ x(t_i) : i = 1, \ldots, N \} \). Let \( \tau_h^0 \) be a regular triangulation of \( \Omega_h \) formed by triangles such that (a) there exists a constant \( C > 0 \) such that for all \( T \in \tau_h^0 \), \( h_T \leq Ch \) (where \( h_T \) is the diameter of \( T \)); (b) any vertex of a triangle lying on the exterior boundary of \( \partial \Omega_h \) belongs to \( \Delta_h \). From \( \tau_h^0 \) we obtain a triangulation \( \tau_h \) of \( \Omega^- \) by replacing each triangle of \( \tau_h^0 \) with one side along the exterior part of \( \partial \Omega_h \) by the corresponding curved triangle.

Let \( T \) be a curved triangle of \( \tau_h \). We denote its vertices by \( a_1^T, a_2^T, \) and \( a_3^T \), numbered in such a way that \( a_1^T \) and \( a_2^T \) are endpoints of the curved side of \( T \). Let \( t_i, t_{i+1} \in [0,1] \) be such that \( x(t_i) = a_2^T \) and \( x(t_{i+1}) = a_3^T \). Then, \( \varphi(t) := x(t_i + t h) \) \( (t \in [0,1]) \) is a parameterization of the curved side of \( T \). Let \( \hat{T} \) be the reference triangle with vertices \( \hat{a}_1 := (0,0), \hat{a}_2 := (1,0), \) and \( \hat{a}_3 := (0,1) \). Consider the affine map \( G_T \) defined by \( G_T(\hat{a}_i) = a_i^T \) for \( i \in \{1,2,3\} \). Consider also the function \( \Theta_T : \hat{T} \to \mathbb{R}^2 \)

\[
\Theta_T(\hat{x}) := \frac{\hat{x}_1}{1 - \hat{x}_2} (\varphi(\hat{x}_2) - (1 - \hat{x}_2)a_2^T - \hat{x}_2a_3^T),
\]

where the limiting value has to be taken as \( \hat{x}_2 \) goes to 1. Then, for \( h \in (0, h_0) \), where \( h_0 \) is sufficiently small, \( T \) is the range of \( \hat{T} \) by the \( C^\infty \) one-to-one mapping \( F_T : \hat{T} \to \mathbb{R}^2 \) given by

\[
F_T := G_T + \Theta_T.
\]

Moreover, each side of \( \hat{T} \) is mapped onto the corresponding side of \( T \), i.e., \( \Theta_T(t,0) = \Theta_T(t,1) = 0 \) and \( F_T(t,1-t) = \varphi(t) \) for all \( t \in [0,1] \). This type of diffeomorphism was first proposed by Zlámal [16] and studied by Scott [12]. If \( T \) is a straight (interior) triangle, we take the curving perturbation \( \Theta_T \equiv 0 \) and thus \( F_T \) is the usual affine map from the reference triangle, this hypothesis will be implicit in the following. Given a regular function \( g : T \to \mathbb{R} \), we denote \( \hat{g} := g \circ F_T \). We also use this notation for vector and matrix-valued functions. When \( T \) is a curved triangle, we need estimates on the derivatives of \( F_T \) and \( F_T^{-1} \) in order to obtain the usual scaling arguments. Such estimates are a consequence of

\[
|\Theta_T|_{k,\infty,T} \leq Ch_T^{\max(2,k)}, \quad k \geq 1,
\]

which is proven in Theorem 22.4 of [15] (cf. also [12]) together with the following results.
Lemma 2. For all $h \in (0, h_0)$, the Jacobian $J_T$ of $F_T$ does not vanish on a neighborhood of $\hat{T}$ and the following estimates hold:

\begin{equation}
C_1 h_T^2 \leq |J_T(\cdot)| \leq C_2 h_T^2,
\end{equation}

\begin{equation}
|B_T|_{k,\infty,\hat{T}} \leq C h_T^{k+1}, \quad |B_T^{-1}|_{k,\infty,\hat{T}} \leq C h_T^{-1} \quad \forall k \geq 0,
\end{equation}

where $B_T := DF_T$.

It follows from (5)–(6) and a careful application of the chain rule that (see [15, Lemma 25.1] and [1])

\begin{equation}
|u|_{m,T} \leq C h_T^{1-m} \|\tilde{u}\|_{m,\hat{T}}, \quad |\tilde{u}|_{m,\hat{T}} \leq C h_T^{m-1} \|u\|_{m,T}
\end{equation}

for all $u$ in $H^m(T)$ and $m \geq 1$.

3.2. Discretization with a curved mixed finite element. On each curved or straight triangle $T$, we define the space

$$P_1(T) := \{p : T \to \mathbb{R} : p \circ F_T \in P_1\},$$

where $P_1$ is the space of polynomials of degree not greater than one. We recall that the barycentric coordinate functions $\lambda_i,T \in P_1(T)$ ($i = 1, 2, 3$) are uniquely determined by $\lambda_i,T(a_j^T) = \delta_{i,j}$. For $1 \leq i \leq 3$, we introduce the functions

$$q_{i,T} := \lambda_{j,T} \lambda_{k,T} \nabla \lambda_{i,T}, \quad (i, j, k) \in C_3 := \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}.$$

Generalizing the mixed finite element of Bernardi–Raugel [2], we take the velocities in the space

$$P(T) := P_1(T) \oplus \text{span}(q_{1,T}, q_{2,T}, q_{3,T}),$$

where $P_1(T) := P_1(T) \times P_1(T)$, and the pressure in the space $P_0$ of constant functions.

It is easy to see that a function $\phi \in P(T)$ is uniquely determined by the nine degrees of freedom given by the Lagrange functionals $N_{i,T}$, $(i = 1, 2, 3)$ (componentwise evaluation at the vertices) plus the moments

$$m_{i,T}(\phi) := \int_{f_i^T} \phi \cdot n_{i,T} d\sigma \quad (i = 1, 2, 3),$$

where $f_i^T$ is the side of $T$ opposite to $a_i^T$ and $n_{i,T}$ is the outward normal to that side. Moreover, if $(i, j, k) \in C_3$, $\phi \in P(T)$, and $\phi(a_i^T) = \phi(a_j^T) = 0$, then $m_{k,T}(\phi) = 0$. Hence, we may define the global spaces corresponding to this mixed finite element by

$$W_h := \{v \in C^0(\Omega^-, \mathbb{R}^2) : v|_T \in P(T), \forall T \in \tau_h\} \cap H^1_T(\Omega^-),$$

and

$$Q_h := \{q \in L^2_T(\Omega^-) : q|_T \in P_0, \forall T \in \tau_h\}.$$
We are now in position to write the discrete problem associated with (3),
\begin{align}
\text{find } (\tau_h, p_h) \in M_h \times Q_h \text{ s.t. }
\begin{align*}
A(\tau_h, \tau) - D(\tau, p_h) &= L(\tau) \quad \forall \tau \in M_h, \\
D(\tau_h, q) &= 0 \quad \forall q \in Q_h,
\end{align*}
\end{align}
(8)

where $M_h := W_h \times H_h$ is the discrete counterpart of $M$.

4. Full discretization of the equations. In this section we give a fully discrete scheme based on the application of numerical integration to the equations of the Galerkin method. We denote a quadrature formula of degree $m$ (i.e., exact for all bivariate polynomials of degree not greater that $m$) on the reference triangle by
\begin{align}
\hat{Q}_m(\hat{\phi}) := \sum_{k=1}^{d_0} \hat{\omega}_k \hat{\phi}(\hat{b}_k) \simeq \int_T \hat{\phi}.
\end{align}
The corresponding quadrature formula on each $T \in \tau_h$ is obtained by a change of variable:
\begin{align}
Q^T_m(\phi) := \hat{Q}_m(|J_T| \hat{\phi}) = \sum_{k=1}^{d_0} \hat{\omega}_k |J_T| (\hat{b}_k) \hat{\phi}(\hat{b}_k) \simeq \int_T \phi.
\end{align}
We remark that the degree of $Q^T_m$ is not the same as that of $\hat{Q}_m$ unless $T$ is a straight triangle. Assuming that $f \in C(\Omega^-)$, we approximate for all $v, u \in W_h$, and $q \in Q_h$
\begin{align}
(f, v)_{0, \Omega^-} \simeq & L_h(v) := \sum_T Q^T_0(f \cdot v), \\
(\nabla \cdot v, q)_{0, \Omega^-} \simeq & d_h(v, q) := \sum_T q_T Q^T_1(\nabla \cdot v),
\end{align}
and
\begin{align}
a(u, v) \simeq a_h(u, v) := 2 \sum_T Q^T_2(E[u] : E[v]).
\end{align}
For the boundary terms, we first consider a basic one-dimensional quadrature formula of degree 2,
\begin{align}
\ell_2(g) := \sum_{k=1}^{d_1} \rho_k g(z_k) \simeq \int_0^1 g,
\end{align}
and introduce, for all $u \in W_h$ and $\mu \in H_h$, the approximation
\begin{align}
(\gamma u, \mu) \simeq (\gamma u, \mu)_h := h \sum_{i=1}^N \mu_i \cdot \ell_2(\gamma u(t_i + h \cdot)),
\end{align}
where $\mu_i$ is the constant value of $\mu$ in $(s_i, s_{i+1})$ and $\ell_2$ is applied componentwise.
Finally, we denote a two-dimensional quadrature formula of degree $m$ on the unit square by
\begin{align}
Z_m(g) := \sum_{k=1}^{d_2} \eta_k g(x_k) \simeq \int_0^1 \int_0^1 g.
\end{align}
With the aid of such a formula we approximate the bilinear form associated with the double-layer potential on $\mathbf{W}_h \times \mathbf{H}_h$ as follows:

$$(K\gamma \mathbf{v}, \mu) \simeq k_h(\gamma \mathbf{v}, \mu) := h^2 \sum_{i,j=1}^{N} \mu_i^\top \mathbf{Z}_2(K_{i,j}),$$

where $K_{i,j}(s,t) := K(t_i + sh, t_j + th)\mathbf{v}(x(t_j + th))$. Here also we apply $\mathbf{Z}_2$ componentwise. Numerical quadratures must be handled with care when defining an approximation of $(V\lambda, \mu)$ on $\mathbf{H}_h \times \mathbf{H}_h$ because of the logarithmic singularity of the kernel $V$. Here, we follow [8] and consider the following decomposition of the kernel:

$$V(s,t) = -\frac{1}{4\pi} \log |s-t| \mathbf{I} + B(s,t).$$

Notice that $B$ is of class $C^\infty$ in the domain $\{(s,t) : |s-t| < 1\}$. Now, the strategy consists of using a formula $\mathbf{Z}_1$ to approximate the second integral and compute the first one exactly (cf. [8] and [5]): i.e.,

$$\int_{s_{i-1}}^{s_i} \int_{s_{j-1}}^{s_j} V(s,t) ds dt \simeq \mathbf{V}_{i,j} := -\frac{1}{4\pi} \log |h| \kappa_{i-j} I + h^2 \mathbf{Z}_1 \left( B(s_i + h \cdot, s_j + h \cdot) \right),$$

with

$$\kappa_k := \int_0^1 \int_0^1 \log |k+t-u| dt du$$

and

$$(i, j) := \begin{cases} (i, j) & \text{if } |i-j| \leq N/2, \\ (i, j-N) & \text{if } i-j > N/2, \\ (i-N, j) & \text{if } j-i > N/2. \end{cases}$$

Notice that the periodicity of $V(\cdot, \cdot)$ allows one to use the indices $(\hat{i}, \hat{j})$ instead of $(i, j)$ and avoid the neighborhood of the region $\{(s,t) : |s-t| = 1\}$. Then we approximate for all $\lambda, \mu \in \mathbf{H}_h$

$$(V\lambda, \mu) \simeq v_h(\lambda, \mu) := \sum_{i,j=1}^{N} \mu_j^\top \mathbf{V}_{i,j} \lambda_i.$$ 

We are now in a position to write a fully discrete method for (3):

$$\begin{align*}
\text{find } & \pi_h^* := (u_h^*, \lambda_h^*) \in \mathcal{M}_h \text{ and } p_h^* \in \mathcal{Q}_h \text{ s.t.} \\
A_h(\pi_h^*, \mathbf{v}) - d_h(\mathbf{v}, p_h^*) &= L_h(\mathbf{v}) \quad \forall \mathbf{v} \in \mathcal{M}_h, \\
d_h(u_h^*, q) &= 0 \quad \forall q \in \mathcal{Q}_h,
\end{align*}$$

where, for all $\pi, \mathbf{v} \in \mathcal{M}_h$,

$$A_h(\pi, \mathbf{v}) := a_h(\mathbf{u}, \mathbf{v}) - (\gamma \mathbf{v}, \lambda)_h + 2v_h(\lambda, \mu) + (\gamma \mathbf{u}, \mu)_h - 2k_h(\gamma \mathbf{u}, \mu).$$

In section 6 we will analyze this family of fully discrete methods. Once the different quadrature formulae are chosen, the numerical scheme is implementable. Adequate choices of these formulae and also efficient strategies for implementation and solution of the (rather complicated) linear system are the aim of forthcoming work.
5. Convergence of the Galerkin method. Let us consider the auxiliary problem:

\[
\begin{align*}
\text{find } (\bar{\pi}, z) & \in M \times L^2_0(\Omega^-) \text{ s.t.} \\
B(\pi, \pi) - D(\pi, z) & = L(\pi) \quad \forall \pi \in M \\
D(\pi, q) & = 0 \quad \forall q \in L^2_0(\Omega^-),
\end{align*}
\]

where \(B(\pi, \pi) := A(\pi, \pi) + (2K\gamma u, \mu)\). This bilinear form is elliptic on \(M\) by virtue of Korn’s inequality and Lemma 1. This fact, together with the inf-sup condition (cf. [7])

\[
\sup_{\bar{\pi} \neq \pi \in \mathcal{M}} \frac{D(\pi, q)}{\|\pi\|_{\mathcal{M}}} \geq \sup_{\theta \neq v \in H^1_0(\Omega^-)} \frac{\langle \nabla \cdot v, q \rangle_{0, \Omega^-}}{\|v\|_{1, \Omega^-}} \geq \beta \|q\|_{0, \Omega^-}, \quad \forall q \in L^2_0(\Omega^-),
\]

prove that the variational problem (10) is well posed. Let us now study the corresponding Galerkin scheme with trial-test space \(M_h \times Q_h\):

\[
\begin{align*}
\text{find } (\bar{\pi}_h, z_h) & \in M_h \times Q_h \text{ s.t.} \\
B(\bar{\pi}_h, \pi) - D(\bar{\pi}_h, z_h) & = L(\pi) \quad \forall \pi \in M_h \\
D(\bar{\pi}_h, q) & = 0 \quad \forall q \in Q_h.
\end{align*}
\]

To this end, we need a discrete counterpart of (11) which is a consequence of the following result.

**Theorem 3.** There exists a linear operator \(\pi_h : H^1_0(\Omega^-) \to W_h\) such that

\[
\langle \nabla \cdot v, q \rangle_{0, \Omega^-} = \langle \nabla \cdot \pi_h v, q \rangle_{0, \Omega^-} \quad \forall q \in Q_h, v \in H^1_0(\Omega^-)
\]

and that the following approximation property holds: for \(m = 0, 1\) and \(k = 1, 2\)

\[
\|v - \pi_h v\|_{m, \Omega^-} \leq Ch^{k-m}k_{k,\Omega^-} \quad \forall v \in H^1_0(\Omega^-) \cap H^k(\Omega^-).
\]

**Proof.** The proof is adapted from that of Lemma 2.2 in [7]. Consider the modification of Clément’s projection for curved elements given in [2], which is a linear operator

\[R_h : H^1(\Omega^-) \to \{ u \in C(\Omega^-) : v | \ggg{T} \in P^1(\ggg{T}), \forall T \ggg{T} \} \cap H^1_0(\ggg{T})\]

satisfying for all \(T \ggg{T}\)

\[
\|v - R_h v\|_{m, \ggg{T}} \leq Ch^{k-m}k_{k,\ggg{T}}
\]

for all \(v \in H^1_0(\ggg{T}) \cap H^k(\ggg{T})\) and for the values of \(m\) and \(k\) given above (cf. [1, Theorem 4.1]). In (15), \(\ggg{T}\) denotes the union of all the triangles surrounding \(T\), i.e., \(\ggg{T} := \{ T' \ggg{T} : T' \ggg{T} \cap T \ggg{T} \neq \emptyset \} \). Let us also denote by \(R_h\) the operator defined on \(H^1_0(\ggg{T})\) by componentwise application of \(R_h\).

Consider now the operator \(\pi_h : H^1_0(\ggg{T}) \to W_h\) given in each triangle \(T\) by the relations

\[
\begin{align*}
\{ \pi_h v(a^T_i) & = R_h v(a^T_i), \quad i = 1, 2, 3, \\
m_{i, T}(\pi_h v) & = m_{i, T}(v), \quad i = 1, 2, 3.
\end{align*}
\]

Notice that for all \(q \in Q_h\),

\[
\int_{\Omega^-} \nabla \cdot (v - \pi_h v) q = \sum_T q | T \int_{\partial T} (v - \pi_h v) \cdot n = 0
\]
by the divergence theorem and the second set of conditions for the definition of $\pi_h$. Hence, (13) holds.

Moreover by construction of $\pi_h$ and since each component of $R_h v$ belongs to $P_1(T)$ we have

$$\pi_h v|_T = R_h v|_T + \sum_{i=1}^{3} \beta_i q_{i,T}|_T$$

with

$$\beta_i = \frac{m_{i,T}(v-R_h v)}{m_{i,T}(q_{i,T})},$$

since $m_{i,T}(q_{j,T}) = 0$ if $i \neq j$ by definition of these functions from the curved barycentric coordinates.

Let $\delta_{i,T}: [0,1] \to \mathbb{R}^2$ be the parameterization of the side $f_{i,T}$ given by

$$\delta_{i,T} := F_T \circ \hat{\delta}_i,$$

$$\hat{\delta}_i(t) := \left\{ \begin{array}{ll}
(1-t,t), & i = 1, \\
(0,t), & i = 2, \\
(t,0), & i = 3.
\end{array} \right.$$ 

It follows readily from (6) that for all $T$

$$C_1 h_T \leq |\delta'_{i,T}(t)| \leq C_2 h_T. \quad (17)$$

On the other hand, we have

$$\hat{q}_{i,T} = \hat{\lambda}_j \hat{\lambda}_k B_T^{-1} \nabla \hat{\lambda}_i, \quad (i,j,k) \in C_3.$$ 

Hence, with $(i,j,k) \in C_3$, we obtain

$$m_{i,T}(q_{i,T}) = \varepsilon_i \int_0^1 \left( \hat{\lambda}_j \hat{\lambda}_k |B_T^{-1}| \nabla \hat{\lambda}_i \right) \circ \hat{\delta}_i(t) |\delta'_{i,T}(t)| \, dt,$$

where $\varepsilon_i = 1$ or $-1$, depending on the coincidence of the directions of the normal and $\nabla \lambda_{i,T}$. Therefore, by (5–6) and (17) we have

$$|m_{i,T}(q_{i,T})| \geq C. \quad (19)$$

Again applying (17) and the trace theorem in the reference triangle $\hat{T}$, it follows that

$$|m_{i,T}(\phi)| \leq C_1 h_T \int_0^1 |\hat{\phi}(\hat{\delta}_i(t))| \, dt \leq C_2 h_T \|\hat{\phi}\|_{1,\hat{T}}. \quad (20)$$

From (19) and (20) we obtain that for $i = 1, 2, 3$

$$|\beta_i| \leq C_1 h_T \|\hat{\phi} - \hat{R}_h \phi\|_{1,\hat{T}} \leq C_2 (\|v - R_h v\|_{0,T} + h_T \|v - R_h v\|_{1,T}) \quad (21)$$

by (7).

Finally, by (18), (7), and Lemma 6, it is easy to prove that for all $T$

$$\|q_{i,T}\|_{m,T} \leq Ch_{T}^{-m}, \quad m = 0, 1. \quad (22)$$
Going back to (16) and applying (21), (22), and (15), we obtain

\begin{equation}
\| \mathbf{v} - \pi_h \mathbf{v} \|_{m,T} \leq \| \mathbf{v} - R_h \mathbf{v} \|_{m,T} + \sum_{i=1}^{3} \| \beta_i \| \| q_{i,T} \|_{m,T} \leq C_{H}^{k-m} \| \mathbf{v} \|_{k,\Delta(T)}.
\end{equation}

The result follows from summing in (23) over all triangles and using the regularity of $\tau_h$. \hfill \Box

**Corollary 4.** There exist $\beta^* > 0$ and $h_0 > 0$ such that for all $h \leq h_0$

\begin{equation}
\inf_{0 \neq q \in Q_h} \left( \sup_{\tau \neq \tau' \in \mathcal{M}_h} \frac{D(\tau, q)}{\| q \|_{0,\Omega} - \| \tau \|_{\mathcal{M}}} \right) \geq \beta^*.
\end{equation}

**Proof.** The result follows from Fortin’s trick by using (11), (13), and the uniform boundedness of $\pi_h$ on $H^1(\Omega^-)$. \hfill \Box

Applying the well-known approximation theory for saddle-point problems (see \cite[p. 114]{14}) we deduce that (12) is well posed and

\begin{equation}
\| \tau - \tau_h \|_{\mathcal{M}} + \| z - z_h \|_{0,\Omega} \leq C \left( \inf_{\tau \in \mathcal{M}_h} \| \tau - \tau \|_{\mathcal{M}} + \inf_{q \in Q_h} \| z - q \|_{0,\Omega} \right).
\end{equation}

Now, let us prove approximation properties of the discrete subspaces. By (14), we have

\begin{equation}
\inf_{\mathbf{v} \in W_h} \| \mathbf{u} - \mathbf{v} \|_{1,\Omega} \leq Ch \| \mathbf{u} \|_{2,\Omega} \quad \forall \mathbf{u} \in H^2(\Omega^-).
\end{equation}

On the other hand, a straightforward property of the $L^2$-orthogonal projection onto $Q_h$ gives

\begin{equation}
\inf_{q \in Q_h} \| p - q \|_{0,\Omega} \leq Ch \| p \|_{1,\Omega} \quad \forall p \in H^1(\Omega^-).
\end{equation}

Finally, it is a consequence of the approximation results in periodic Sobolev spaces (see \cite{14}) that

\begin{equation}
\inf_{\mu_h \in H_h} \| \mu - \mu_h \|_{-\frac{1}{2}} \leq Ch \| \mu \|_{\frac{1}{2}} \quad \forall \mu \in H^{\frac{1}{2}} \cap H^{\frac{1}{2}}_{0,\Omega^-}.
\end{equation}

Therefore, in the general case, by density of smooth functions in $\mathcal{M} \times L^2_0(\Omega^-)$, the previous inequalities yield

\begin{equation}
\lim_{h \to 0} \left( \inf_{\tau \in \mathcal{M}_h} \| \tau - \tau \|_{\mathcal{M}} + \inf_{q \in Q_h} \| p - q \|_{0,\Omega} \right) = 0.
\end{equation}

Hence, from (24) and (25), it is clear that the discrete scheme (12) is convergent.

**Theorem 5.** For all $h$ small enough, problem (8) has a unique solution. Moreover, the Galerkin method is stable and we have Céa’s estimate

\begin{equation}
\| \tau - \tau_h \|_{\mathcal{M}} + \| p - p_h \|_{0,\Omega} \leq C \left( \inf_{\tau \in \mathcal{M}_h} \| \tau - \tau \|_{\mathcal{M}} + \inf_{q \in Q_h} \| p - q \|_{0,\Omega} \right).
\end{equation}

In case the exact solution belongs to $H^2(\Omega^-) \times H^{1/2} \times H^1(\Omega^-)$ we have

\begin{equation}
\| \mathbf{u} - \mathbf{u}_h \|_{1,\Omega} + \| p - p_h \|_{0,\Omega} + \| \lambda - \lambda_h \|_{-\frac{1}{2}} \leq Ch \left( \| \mathbf{u} \|_{2,\Omega} + \| p \|_{1,\Omega} + \| \lambda \|_{\frac{1}{2}} \right).
\end{equation}
Proof. The theorem is a consequence of a classical result for compact perturbations of operator equations. Indeed, both (3) and (10) are well-posed, (3) is a compact perturbation of (10) (see Lemma 1), and (12) is convergent. Under these hypotheses, Theorem 13.7 of [10] shows that, if $h$ is sufficiently small, (8) is also well posed and convergent. Finally, Theorem 13.6 of [10] shows that the convergence implies Céa’s estimate (26). The last assertion of the theorem follows from the approximation properties of the discrete subspaces given above. □

6. Analysis of the fully discrete method. In this section we study the stability and convergence of the fully discrete method defined in section 4. First we will give some bounds concerning the six kinds of approximations. With these, we will be able to prove a uniform inf-sup condition and, hence, convergence of the perturbed Galerkin scheme.

6.1. Technical results.

Lemma 6. For all $T \in \tau_h$ and $m \geq 0$

$$|B^{-1}_T|_{m,\hat{T}} \leq Ch_T^{m-1}.$$  

Proof. Consider the following matrix operator

$$A^* := (\text{det } A) A^{-1} = \begin{pmatrix} A_{2,2} & -A_{1,2} \\ -A_{2,1} & A_{1,1} \end{pmatrix}.$$  

Then $B^{-1}_T = (1/J_T)B^*_T$. By (5)–(6) it is easy to prove the estimate

$$|J^k_T|_{m,\hat{T}} \leq Ch_T^{m+2k}, \quad m \geq 0, \quad k \in \mathbb{Z}.$$  

Thus the statement of the Lemma is a straightforward consequence of Leibniz’s rule, (28) and (6). □

Lemma 7. There exist $h_0 > 0$ and $C > 0$ such that for any curved triangle $T \in \tau_h$

$$|\hat{r}_{1,\hat{T}}| \leq C \left( |B_T \hat{p} + \hat{r}_{1,\hat{T}}| + h_T \|B_T \hat{p} + \hat{r}_{1,\hat{T}}\|_0,\hat{T} \right) \quad (\forall h \leq h_0)$$  

for all $\hat{p} \in P_1(\hat{T})$ and $\hat{r} \in \text{span}(q_1,\hat{T},q_2,\hat{T},q_3,\hat{T})$.

Proof. Let us first prove that

$$\|DG_T \hat{p}\|_{0,\hat{T}} + \|\hat{r}\|_{0,\hat{T}} \leq C\|B_T \hat{p} + \hat{r}\|_{0,\hat{T}} \quad (\forall h_T \leq h_1).$$  

The equivalence of the norms $\hat{p} + \hat{r} \to \|\hat{p}\|_{0,\hat{T}} + \|\hat{r}\|_{0,\hat{T}}$ and $\hat{p} + \hat{r} \to \|\hat{p} + \hat{r}\|_{0,\hat{T}}$ on $P(\hat{T}) := P_1(\hat{T}) \oplus \text{span}(q_1,\hat{T},q_2,\hat{T},q_3,\hat{T})$ and the triangle inequality give

$$\|DG_T \hat{p}\|_{0,\hat{T}} + \|\hat{r}\|_{0,\hat{T}} \leq \tilde{C}\|DG_T \hat{p} + \hat{r}\|_{0,\hat{T}} \leq \tilde{C} \left( \|B_T \hat{p} + \hat{r}\|_{0,\hat{T}} + \|DG_T \hat{p}\|_{0,\hat{T}} \right),$$  

since $DG_T$ is a constant matrix and $B_T := DG_T + D\Theta_T$. Now (4) and the fact that $DG_T^{-1}$ is bounded by $C_0h_T^{-1}$ provide the estimate

$$\|DG_T \hat{p}\|_{0,\hat{T}} + \|\hat{r}\|_{0,\hat{T}} \leq C \left( \|B_T \hat{p} + \hat{r}\|_{0,\hat{T}} + \|DG_T^{-1}\|_{0,\hat{T}} \right),$$  

$$\|DG_T \hat{p}\|_{0,\hat{T}} + \|\hat{r}\|_{0,\hat{T}} \leq C \left( \|B_T \hat{p} + \hat{r}\|_{0,\hat{T}} + h_T \|DG_T \hat{p}\|_{0,\hat{T}} \right).$$
and (29) follows. Similar arguments show that we also have
\[ |DG_T \hat{p}|_{1, \hat{T}} + |\hat{r}|_{1, \hat{T}} \leq \|DG_T \hat{p} + \hat{r}\|_{1, \hat{T}} \leq C_0 \|DG_T \hat{p} + \hat{r}\|_{1, \hat{T}}. \]

Notice that the left-hand side of the last inequality remains invariant if we add a constant vector to \(DG_T \hat{p}\). Therefore, we also have
\[ |DG_T \hat{p}|_{1, \hat{T}} + |\hat{r}|_{1, \hat{T}} \leq C|DG_T \hat{p} + \hat{r}|_{1, \hat{T}} \leq C \left(|B_T \hat{p} + \hat{r}|_{1, \hat{T}} + |D\Theta_T \hat{p}|_{1, \hat{T}}\right), \]
where the last step is simply the triangle inequality. Now, using Leibniz’s rule together with (4) and (29) we obtain
\[ |D\Theta_T \hat{p}|_{1, \hat{T}} \leq C_1 \left(\|D\Theta_T DG_T^{-1}\|_{0, \infty, \hat{T}} \|DG_T \hat{p}\|_{1, \hat{T}} + \|D\Theta_T DG_T^{-1}\|_{1, \infty, \hat{T}} \|DG_T \hat{p}\|_{0, \hat{T}}\right) \leq C_2 h_T \left(|DG_T \hat{p}|_{1, \hat{T}} + \|B_T \hat{p} + \hat{r}\|_{0, \hat{T}}\right), \]
and the result follows. \( \Box \)

**Lemma 8.** There exist \( h_0 > 0 \) and \( C > 0 \) such that for all \( h \leq h_0 \) and for all \( T \in \tau_h \)
\[ \|v\|_{2, T} \leq Ch_0^{-1} \|v\|_{1, T} \quad \forall v \in P(T). \]

**Proof.** The statement follows from standard arguments in the case of straight triangles. Let \( T \in \tau_h \) be a curved triangle. The chain rule and the properties given in Lemma 2 permit one to obtain the estimate
\[ |v|_{2, T} = |q|_{2, T} \leq C_1 \left(h_T^{-1} |\hat{q}|_{2, \hat{T}} + h_T |\hat{q}|_{1, \hat{T}}\right), \]
for all \( v := p + q \in P(T) \), where \( p \) is the component of \( v \) that belongs to \( P_1(T) \). Notice that \( \hat{q} := B_T^{-1}r \) for a function \( \hat{r} \in \operatorname{span}(q_1, \ldots, q_3) \) (see (18)). Hence, using Leibniz’s rule together with Lemma 6, we obtain
\[ |v|_{2, T} \leq C_2 \left(h_T^{-2} |\hat{r}|_{2, \hat{T}} + h_T^{-1} |\hat{r}|_{1, \hat{T}} + \|\hat{r}\|_{0, \hat{T}}\right) \leq C_3 \left(h_T^{-2} |\hat{r}|_{1, \hat{T}} + \|\hat{r}\|_{1, \hat{T}}\right), \]
where the last inequality follows from the fact that \( |\hat{q}|_{2, \hat{T}} \leq c_0 |\hat{q}|_{1, \hat{T}} \) for all polynomial functions \( \hat{q} \) of degree not greater than 2. We apply Lemma 7 and (29) (notice that \( B_T \hat{v} = B_T \hat{p} + \hat{r} \)) to deduce that, for sufficiently small \( h_T \),
\[ |v|_{2, T} \leq C_3 \left(h_T^{-2} |B_T \hat{v}|_{1, \hat{T}} + h_T^{-1} \|B_T \hat{v}\|_{0, \hat{T}}\right) \leq C_4 \left(h_T^{-1} |\hat{v}|_{1, \hat{T}} + \|\hat{v}\|_{0, \hat{T}}\right), \]
where in the last inequality we again used Leibniz’s rule and (6). Finally, the result follows by changing back to \( T \). \( \Box \)

**Corollary 9.** There exists \( C > 0 \) such that for any \( T \in \tau_h \)
\[ \|p_T\|_{1, T} + h_T^{-1} \max_{1 \leq i \leq 3} |\alpha_i T| \leq C \|v\|_{1, T} \quad \forall v \in P(T), \]
where
\[ v = p_T + \sum_{i=1}^{3} \alpha_i T q_i, \]
is the canonical decomposition of \( v \) with \( p_T \in P_1(T) \).
Proof. On the one hand, identity \( m_{i,T}(q_{i,T})\alpha_i = m_{i,T}(v - p_T) \) and estimates (19) and (20) lead to

\[ |\alpha_i| \leq Ch_T \|\hat{v} - \hat{p}\|_{1,T}. \]

But \( \hat{p} \) interpolates \( \hat{v} \) at the three vertices of \( \hat{T} \). Hence, it follows from the Bramble–Hilbert lemma, (6), and Lemma (8) that

\[ |\alpha_{i,T}| \leq Ch_T \|v\|_{1,T}. \]

On the other hand,

\[ \|p_T\| \leq \|v\|_{1,T} + Ch_T \max_{1 \leq i \leq 3} \|q_{i,T}\|_{1,T}, \]

and then the result is a consequence of (22).

6.2. Interior terms. Let us consider the error functionals

\[ \hat{E}_m := \hat{Q}_m - \int_{\hat{T}} \quad E_{i,m}^T := Q_{i,m}^T - \int_T = \hat{E}_m[|J_T|^\gamma]. \]

Notice that \( \hat{E}_m \) satisfies

\[ |\hat{E}_m[\phi]| \leq C \|\phi\|_{0,\infty,\hat{T}} \quad \forall \phi \in C(\hat{T}). \]

Then we have this direct extension of the Bramble–Hilbert Lemma, which will be of use in what follows.

Lemma 10. Let \( n, k \geq 1 \) and let

\[ \mathcal{X} : V_n^1 \times \cdots \times V_n^1 \times V_k^1 \times \cdots \times V_k^2 \to \mathbb{R} \]

be a \((n+k)\)-linear functional with \( V_i^j \) finite dimensional real spaces (in our applications \( \mathbb{R}, \mathbb{R}^2, \) or \( \mathbb{R}^{2\times 2} \)). If \( a_i : \hat{T} \to V_i^j \) are polynomials such that \( \sum_{i=1}^n \deg a_i \leq m \), then

\[ |\hat{E}_m[\mathcal{X}(a_1, \ldots, a_n, f_1, \ldots, f_k)]| \leq C \sum_{i=1}^k |f_i|_{1,\infty,\hat{T}} \left( \prod_{j \neq i} \|f_j\|_{0,\infty,\hat{T}} \right) \]

for every \( f_i \in W^{1,\infty}(\hat{T}, V_i^2) \).

Proposition 11. If \( f \in W^{1,\infty}(\Omega^-) \), then there exists a constant \( C \) such that

\[ |(f, v)_{0,\Omega^-} - L_h(v)| \leq Ch\|f\|_{1,\infty,\Omega^-}\|v\|_{1,\Omega^-} \quad \forall v \in \mathcal{W}_h. \]

Proof. Decomposing the error into the triangles, using (30) and Lemma 9, we obtain

\[ |(f, v)_{0,\Omega^-} - L_h(v)| \leq \sum_T |E_{i,T}^T [f \cdot p_T]| + C \sum_T h_T \|v\|_{1,T} \left( \max_i |E_{i,T}^T [f \cdot q_{i,T}]| \right). \]

We remark first that for \( m = 0, 1 \)

\[ |f|_{m,\infty,\hat{T}} \leq Ch_T^m |f|_{m,\infty,T}. \]
Applying then Lemma 10 and the bounds for the Jacobian ((28) with \( k = 1 \)) we obtain for all \( p \in P_1(T) \)

\[(33) \quad |E_0^T[f \cdot p]| = |\tilde{E}_0[J_T \tilde{f} \cdot \tilde{p}]| \leq Ch_T^2 \|f\|_{1,\infty,T} \|p\|_{1,T},\]

since

\[(34) \quad |\tilde{p}|_{m,\infty,\hat{T}} \leq Ch_T^{m-1} \|p\|_{m,T}\]

due to the fact that \( \tilde{p} \in P_1 \) (see also [15, Theorem 11.6 and Lemma 22.7]). On the other hand, for \( i = 1, 2, 3, \)

\[(35) \quad |E_0^T[f \cdot q_i,T]| \leq Ch_T^2 \|\tilde{f}\|_{0,\infty,\hat{T}} \|\tilde{q}_i,T\|_{0,\infty,\hat{T}} \leq Ch_T \|f\|_{0,\infty,T}\]

by (5), (31), and the fact that \( \|q_i,T\|_{0,\infty,T} \leq Ch_T^{-1}. \) Then (32), (33), (35), and Lemma 9 prove

\[|(f, v)_{0,\Omega} - L_h(v)| \leq Ch \|f\|_{1,\infty,\Omega} \sum_T h_T \|v\|_{1,T},\]

from where the result follows by applying the fact that \( \sum_T h_T^2 \leq C. \)

**Proposition 12.** There exists a constant \( C \) such that

\[|\nabla \cdot (v, q)_{0,\Omega} - d_h(v, q)| \leq Ch \|v\|_{1,\Omega} - q\|_{0,\Omega} - \forall v \in W_h, \quad \forall q \in Q_h.\]

**Proof.** If \( q_T \) is the constant value of \( q \) on \( T \) and we apply decomposition (30) and Lemma 9, it follows that

\[(36) \quad |(\nabla \cdot v, q)_{0,\Omega} - d_h(v, q)| \leq \sum_T \left| q_T \left(E_0^T[\nabla \cdot p_T] + Ch_T \|v\|_{1,T} \max_i |E_0^T[\nabla \cdot q_i,T]| \right) \right| .\]

We will prove that for all \( T, p \in P_1(T), \) and \( i = 1, 2, 3, \)

\[(37) \quad |E_0^T[\nabla \cdot p]| \leq Ch_T^2 \|p\|_{1,T},\]

\[(38) \quad |E_0^T[\nabla \cdot q_i,T]| \leq Ch_T.\]

Then the statement of the proposition is proven by (36), the Cauchy–Schwarz inequality, Lemma 9, and the fact that

\[\sum_T h_T^2 |q_T|^2 \leq C \|q\|^2_{0,\Omega} - \forall q \in Q_h.\]

Recall the definition of the operator \(*\) in (27), so that \( J_T B_T^{*-1} = B_T^* \), and the fact that the sign of \( J_T \) remains constant on \( \hat{T} \). Then if \( X_1(\lambda|A_1, A_2) := \lambda \text{tr}(A_2^* A_1), \) with \( \text{tr}(A) = \sum_i A_{i,i} \), we have

\[|E_0^T[\nabla \cdot p]| = |\tilde{E}_0[\nabla \cdot \tilde{p}]| = |\tilde{E}_0[X_1(1) D\tilde{p}, B_T]|.\]

Then, by Lemma 10, (6), and (34) we have for all \( p \in P_1(T) \)

\[|E_0^T[\nabla \cdot p]| \leq C \|D\tilde{p}\|_{0,\infty,\hat{T}} |B_T|_{1,\infty,\hat{T}} \leq Ch_T^2 \|p\|_{1,T},\]

i.e., (37).
We now prove (38). If \((i, j, k) \in C_3\), we have
\[
\nabla \cdot \mathbf{q}_{i,T} = \nabla(\lambda_{j,T} \lambda_k,T) \cdot \nabla \lambda_i,T + \lambda_{j,T} \lambda_k,T \Delta \lambda_i,T.
\]
Since \(J_T \nabla \tilde{\phi} = B_T^* \nabla \tilde{\phi}\), we obtain
\[
|E_T^T \nabla(\lambda_{j,T} \lambda_k,T) \cdot \nabla \lambda_i,T| = \left| \tilde{E}_1 \mathcal{A}_2(\nabla(\lambda_{j,T} \lambda_k), \nabla \lambda_i | 1/J_T, B_T, B_T) \right|,
\]
where \(\mathcal{A}_2(a_1, a_2|\lambda, A_1, A_2) := \lambda(A_1^* a_1) \cdot (A_2^* a_2)\). From (40), Lemma 10, (6), and (28) we then obtain
\[
|E_T^T \nabla(\lambda_{j,T} \lambda_k,T) \cdot \nabla \lambda_i,T| \leq C h_T.
\]
For the second term in (39) we have the equality
\[
\Delta \lambda_i,T \circ \mathbf{F}_T = \nabla \lambda_i \cdot (\Delta \mathbf{F}_T^{-1}) \circ \mathbf{F}_T
\]
since the second derivatives of \(\lambda_i\) are null. Therefore, by (31) and the usual bounds (5)–(6), it follows that
\[
|E_T^T [\lambda_{j,T} \lambda_k,T \Delta \lambda_i,T]| \leq C h_T^2 \|\Delta \mathbf{F}_T^{-1}\| \circ \mathbf{F}_T \|_{0,\infty, \tilde{T}} \leq C h_T.
\]
Then (38) is a straightforward consequence of (39), (41), and (42).

**Proposition 13.** There exists a constant \(C\) such that
\[
|a(\mathbf{u}, \mathbf{v}) - a_h(\mathbf{u}, \mathbf{v})| \leq C h \|\mathbf{u}\|_{1, \Omega^{-}} \|\mathbf{v}\|_{1, \Omega^{-}} \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{W}_h.
\]

**Proof.** If we decompose
\[
\mathbf{u}|_T = \mathbf{p}_T + \sum_{i=1}^{3} \alpha_{i,T} \mathbf{q}_{i,T}, \quad \mathbf{v}|_T = \tilde{\mathbf{p}}_T + \sum_{i=1}^{3} \beta_{i,T} \mathbf{q}_{i,T},
\]
then
\[
\frac{1}{2} |a(\mathbf{u}, \mathbf{v}) - a_h(\mathbf{u}, \mathbf{v})| \leq \sum_T \left| E_T^T \mathbf{E} \mathbf{p}_T : \mathbf{E} \tilde{\mathbf{p}}_T \right| + C \sum_T h_T \|\mathbf{u}\|_{1,T} \max_i \left| E_T^T \mathbf{E} \mathbf{q}_{i,T} : \mathbf{E} \tilde{\mathbf{p}}_T \right| + C \sum_T h_T \|\mathbf{v}\|_{1,T} \max_i \left| E_T^T \mathbf{E} \mathbf{q}_{i,T} : \mathbf{E} \mathbf{p}_T \right| + C \sum_T h_T^2 \|\mathbf{u}\|_{1,T} \|\mathbf{v}\|_{1,T} \max_{i,j} \left| E_T^T \mathbf{E} \mathbf{q}_{i,T} : \mathbf{E} \mathbf{q}_{j,T} \right|.
\]
Hence, if we prove that for all \(T \in \tau_h\), \(\mathbf{p}, \tilde{\mathbf{p}} \in \mathbf{P}_1(T)\), and \(i, j \in \{1, 2, 3\}\)
\[
|E_T^T \mathbf{E} \mathbf{p} : \mathbf{E} \tilde{\mathbf{p}}| \leq C h_T \|\mathbf{p}\|_{1,T} \|\tilde{\mathbf{p}}\|_{1,T},
\]
\[
|E_T^T \mathbf{E} \mathbf{q}_{i,T} : \mathbf{E} \mathbf{p}| \leq C \|\mathbf{p}\|_{1,T},
\]
\[
|E_T^T \mathbf{E} \mathbf{q}_{i,T} : \mathbf{E} \mathbf{q}_{j,T}| \leq C h_T^{-1},
\]
the result is as usual a consequence of Lemma 9.
We first remark that $J_T \tilde{D}w = B_T^{-1} D \tilde{w}$ (with * defined in (27)). Thus, defining

$$\chi_3(\lambda, A_1, A_2, A_3, A_4) := \frac{\lambda}{4} (A_3^* A_1 + (A_3^* A_1)^\top) : (A_4^* A_2 + (A_4^* A_2)^\top)$$

we have

$$|E_2^T[E[a] : E[b]]| = |\tilde{E}_2[\chi_3(1/J_T, D\tilde{a}, D\tilde{b}, B_T, B_T)]|.$$

Taking $p, \tilde{p} \in P_1(T)$, and applying (28) and (6), it follows that

$$|E_2^T[E[p] : E[\tilde{p}]]| \leq C h_T \|D\tilde{p}\|_{0, \infty, \tilde{T}} \|D\tilde{p}\|_{0, \infty, \tilde{T}}.$$

Then (43) is a consequence of (34).

Since for $(i, j, k) \in C_3$

$$Dq_{i,T} = \frac{1}{J_T} \nabla (\tilde{\lambda}_j \tilde{\lambda}_k) \left( B_T^* \nabla \tilde{\lambda}_i ^\top \right) + \tilde{\lambda}_j \tilde{\lambda}_k D \left( B_T^{-1} \nabla \tilde{\lambda}_i ^\top \right)$$

(notice that the second term appears only in curved triangles), we have for all $p \in P_1(T)$,

$$|E_2^T[E[p] : E[q_{i,T}]| \leq \left| \tilde{E}_2[\chi_3(1/J_T^2, D\tilde{p}, \nabla (\tilde{\lambda}_j \tilde{\lambda}_k) \left( B_T^* \nabla \tilde{\lambda}_i ^\top \right), B_T, B_T)] \right| + \left| \tilde{E}_2[\chi_3(1/J_T^2, D\tilde{p}, \tilde{\lambda}_j \tilde{\lambda}_k D \left( B_T^{-1} \nabla \tilde{\lambda}_i ^\top \right), B_T, B_T)] \right|.$$

For the first term we apply Lemma 10 to

$$\chi_1(A_0|\lambda, A_1, A_2, A_3, A_4) := \chi_3(\lambda, A_1, A_0 A_2^\top, A_3, A_4),$$

i.e., taking $A_0 = \nabla (\tilde{\lambda}_j \tilde{\lambda}_k)(\nabla \tilde{\lambda}_i)^\top$ as polynomial part. Then we use estimates (28) and (6) in order to bound this term by $C\|p\|_{1, T}$. By Lemma 6 it follows that

$$\left\| \tilde{\lambda}_j \tilde{\lambda}_k D \left( B_T^{-1} \nabla \tilde{\lambda}_i ^\top \right) \right\|_{0, \infty, \tilde{T}} \leq C \|B_T^{-1}\|_{1, \infty, \tilde{T}} \leq C'.$$

Thus, for the second term we simply apply (31), (47), and the usual bounds for the remaining functions. Therefore (44) is proven.

Finally, if $(i, j, k), (l, m, n) \in C_3$, and we use (46), we obtain

$$J_T E[q_{i,T}] : E[q_{l,T}] = \chi_3(1/J_T^2, \nabla (\tilde{\lambda}_j \tilde{\lambda}_k) \left( B_T^* \nabla \tilde{\lambda}_i ^\top \right), \nabla (\tilde{\lambda}_m \tilde{\lambda}_n) \left( B_T^* \nabla \tilde{\lambda}_l ^\top \right), B_T, B_T)$$

$$+ \chi_3(1/J_T^2, \nabla (\tilde{\lambda}_j \tilde{\lambda}_k) \left( B_T^* \nabla \tilde{\lambda}_i ^\top \right), \tilde{\lambda}_m \tilde{\lambda}_n D \left( B_T^{-1} \nabla \tilde{\lambda}_l ^\top \right), B_T, B_T)$$

$$+ \chi_3(1/J_T^2, \tilde{\lambda}_j \tilde{\lambda}_k D \left( B_T^{-1} \nabla \tilde{\lambda}_i ^\top \right), \nabla (\tilde{\lambda}_m \tilde{\lambda}_n) \left( B_T^* \nabla \tilde{\lambda}_l ^\top \right), B_T, B_T)$$

$$+ \chi_3(1/J_T^2, \tilde{\lambda}_j \tilde{\lambda}_k D \left( B_T^{-1} \nabla \tilde{\lambda}_i ^\top \right), \tilde{\lambda}_m \tilde{\lambda}_n D \left( B_T^{-1} \nabla \tilde{\lambda}_l ^\top \right), B_T, B_T)$$

$$=: T_1 + T_2 + T_3 + T_4.$$

The first term is bounded by again using Lemma 10 since

$$T_1 = \chi_3(\nabla (\tilde{\lambda}_j \tilde{\lambda}_k)(\nabla \tilde{\lambda}_l)^\top, \nabla (\tilde{\lambda}_m \tilde{\lambda}_n)(\nabla \tilde{\lambda}_l)^\top|1/J_T^3, B_T, B_T, B_T), B_T).$$
with

$$X_5(A_1, A_2; \lambda, B_1, B_2, B_3, B_4) := X_3(\lambda, A_1 B_1^T, A_2 B_2^T, B_3, B_4).$$

The last three terms are estimated as follows by (31):

$$h_T |E_2[T_2 + T_3]| + |E_2[T_4]| \leq C.$$

Hence, (45) is proven.

6.3. **Boundary terms.** The analysis of boundary integral terms will include the treatment of the function $\gamma v$ for $v \in W_h$. We begin with some notations. Considering the boundary points \( \{ x(t_i) : i = 1, \ldots, N \} \), we can give a numbering of all the curved triangles, \( \{ T_1, \ldots, T_N \} \) in such a way that \( x(t_i) \) and \( x(t_{i+1}) \) are the vertices of the curved side of \( T_i \). Then let \( v_i : [0, 1] \to \mathbb{R}^2 \) be given by

(48) \[ v_i := v(x(t_i + \cdot h)) = \gamma v(t_i + \cdot h). \]

We will also denote

$$\|g\|_{\infty} := \max_{0 \leq t \leq 1} |g(t)|.$$

We collect in the next lemma some results that will be useful in what follows.

**Lemma 14.** For all \( v \in W_h \) we have the following set of decompositions:

(49) \[ v_i(t) = p_i(t) + \alpha_i p(t) g_i(t), \]

where \( p(t) := t(1 - t) \), \( p_i \) is a polynomial of degree one and for all \( i \in \{1, \ldots, N\} \)

(50) \[ |\alpha_i| \leq Ch \|v\|_{1, T_i}, \]

(51) \[ h \|g\|_{\infty} + \|g'_\|_{\infty} \leq C, \]

(52) \[ h \|p\|_{\infty} \leq C \|v\|_{1, T_i}. \]

**Proof.** By (30) applied at \( T_i \), we have (49) with

$$p_i(t) := p_{T_i}(x(t_i + th)), \quad \alpha_i := \alpha_{1, T_i}, \quad g_i(t) := \nabla \hat{\lambda}_{1, T_i}(t, 1 - t).$$

This follows from the construction of the finite elements, the fact that \( q_{2, T_i} \) and \( q_{3, T_i} \) are identically null on the curved side of \( T_i \), and the form of the barycentric coordinates at the reference triangle. Then (50) follows from Lemma 9. To prove (51) notice simply that

$$g_i(t) = (B_{T_i}^{-1}(t, 1 - t)) \nabla \hat{\lambda}_1$$

and apply Lemma 6.

Since \( \tilde{p}_{T_i} \in P_1 \), we have the bounds

$$\|p\|_{\infty} \leq \|p_{T_i}\|_{0, \infty, T_i} = \|\tilde{p}_{T_i}\|_{0, \infty, \hat{T}} \leq Ch_{T_i}^{-1} \|p_{T_i}\|_{0, T_i}.$$

Then (52) follows from Lemma 9. \( \Box \)
PROPOSITION 15. There exists a constant $C$ such that

$$|(V \lambda, \mu) - v_h(\lambda, \mu)| \leq C h \frac{\|\lambda\|}{\|\mu\|} \cdot \frac{1}{2} \quad \forall \lambda, \mu \in H_h.$$  

Proof. It is a simple adaptation of Lemma 11 in [5]. □

We introduce the following error functionals:

$$e_2 := \ell_2 - \int_0^1, \quad E_2 := Z_2 - \int_0^1 \int_0^1.$$  

PROPOSITION 16. There exists a constant $C$ such that

$$|\gamma v, \mu) - (\gamma v, \mu)| \leq C h \|v\|_{1, \Omega} \|\mu\|_{-\frac{1}{2}} \quad \forall v \in W_h, \quad \forall \mu \in H_h.$$  

Proof. With the definition of $v_i$ in mind (cf. (48)) we have

$$(\gamma v, \mu) - (\gamma v, \mu) = h \sum_{i=1}^N \mu_i \cdot e_2[v_i] = h \sum_{i=1}^N \alpha_i \mu_i \cdot e_2[p g_i],$$

since $v_i = p_i + \alpha_i p g_i$ with $p_i$ componentwise linear and $\ell_2$ of degree 2. This fact implies also that $e_2[p g_i]$ is invariant if we add a constant vector to $g_i$. Hence the Bramble–Hilbert lemma provides

$$|e_2[p g_i]| \leq C \|g_i\|_{\infty}.$$  

Using this last estimate and the bounds given in Lemma 14, we deduce that

$$|(\gamma v, \mu) - (\gamma v, \mu)| \leq |C h^2 \|v\|_{1, \Omega} \|\mu\|_{-\frac{1}{2}} \leq C h^{3/2} \left( \sum_{i=1}^N \|v\|_{1, T_i}^2 \right)^{1/2} \left( \sum_{i=1}^N h |\mu_i|^2 \right)^{1/2},$$

and the result follows from the inverse inequality (cf. [4])

$$\left( \sum_{i=1}^N |\mu_i|^2 \right)^{1/2} \leq \|\mu\|_0 \leq C h^{-1/2} \|\mu\|_{-\frac{1}{2}} \quad \forall \mu \in H_h. \quad \square$$

PROPOSITION 17. There exists a constant $C$ such that

$$(K \gamma \gamma v, \mu) - k h (\gamma v, \mu) \leq C h \|v\|_{1, \Omega} \|\mu\|_{-\frac{1}{2}} \quad \forall v \in W_h, \quad \forall \mu \in H_h.$$  

Proof. The decomposition (48) induces a natural splitting of the error:

$$(K \gamma \gamma v, \mu) - k h (\gamma v, \mu) = h^2 \sum_{i,j=1}^N \mu_i \cdot E_2[\mathbf{K}_{i,j}^1] + h^2 \sum_{i,j=1}^N \alpha_i \mu_i \cdot E_2[\mathbf{K}_{i,j}^2],$$

where

$$\mathbf{K}_{i,j}^1(s, t) := \mathbf{K}(t_i + s h, t_j + th)p_j(t), \quad \mathbf{K}_{i,j}^2(s, t) := p(t)\mathbf{K}(t_i + s h, t_j + th)g_j(t).$$

Now, $E_2[\mathbf{K}_{i,j}^1]$ and $E_2[\mathbf{K}_{i,j}^2]$ are bounded separately by using the fact that $\mathbf{Z}_2$ is of degree 2 and the Bramble–Hilbert lemma. Indeed, we notice that $E_2[\mathbf{K}_{i,j}^1]$ is invariant if we add a matrix with linear components to $\mathbf{K}(t_i + s h, t_j + th)$. Thus, it is straightforward that

$$|E_2[\mathbf{K}_{i,j}^1]| \leq C |\mathbf{K}(t_i + s h, t_j + th)|_{2, \infty, D} \|p_j\|_{\infty} \leq C' h^2 \|p_j\|_{\infty},$$

with $C'$ independent of $h$. □
where we denoted $D := (0, 1) \times (0, 1)$. In the same way, $E_2[K^2_{i,j}]$ remains invariant when adding a constant vector to $K(t_i + sh, t_j + th)g_j(t)$ and therefore

$$|E_2[K^2_{i,j}]| \leq |K(t_i + sh, t_j + th)g_j(t)|_{1,\infty,D} \leq C'h\|g_j\|_{\infty} + C'\|g'_j\|_{\infty},$$

where the last inequality follows from Leibniz’s rule. The last estimates together with (50–52) give

$$|(K\gamma v, \mu) - k_h(\gamma v, \mu)| \leq C h^3 \sum_{i=1}^{N} |\mu_i| \sum_{j=1}^{N} \|v\|_{1,T_j},$$

and we conclude as in the last proposition by the Cauchy–Schwarz inequality and (53).


**Theorem 18.** Assume that $f \in W^{1,\infty}(\Omega^{-})$. Then for $h$ small enough the fully discrete scheme given by (9) is well posed and convergent. Moreover, we have

$$\|u - u_h^n\|_{1,\Omega^{-}} + \|p - p_h^n\|_{0,\Omega^{-}} + \|\lambda - \lambda_h^n\|_{-\frac{1}{2},\Omega^{-}} \leq Ch(||f||_{1,\infty,\Omega^{-}} + \|u\|_{2,\Omega^{-}} + \|p\|_{1,\Omega^{-}} + ||\lambda||_{\frac{1}{2}}),$$

in case the exact solution belongs to $H^2(\Omega^{-}) \times H^{1/2} \times H^1(\Omega^{-}).$

**Proof.** Let $S_h := W_h \times H_h \times Q_h$ and $S := H^1(\Omega^{-}) \times L^2(\Omega^{-})$ with the obvious definition for the product norm $\|\cdot\|_{S}$. Then we can write problems (2) and (8) in a condensed form: find $\xi \in S$ and $\xi_h \in S_h$ such that

$$A(\xi, \eta) = L(\eta) \quad \forall \eta \in S, \quad A(\xi_h, \eta) = L(\eta) \quad \forall \eta \in S_h.$$  

From the analysis of the Galerkin method (see Theorem 5) we have the uniform inf-sup condition of $A$ on $S_h$

$$\sup_{0 \neq \eta \in S_h} \frac{A(\chi, \eta)}{\|\eta\|_S} \geq \beta\|\chi\|_S \quad \forall \chi \in S_h.$$  

The fully discrete scheme (9) can also be written by means of an approximate family of bilinear forms $A_h : S_h \times S_h \rightarrow \mathbf{R}$ and linear functionals $L_h : S_h \rightarrow \mathbf{R}$: find $\xi_h^n \in S_h$ such that

$$A_h(\xi_h^n, \eta) = L_h(\eta) \quad \forall \eta \in S_h.$$  

Obviously, Propositions 12–17 have proven that

$$|A(\chi, \eta) - A_h(\chi, \eta)| \leq C h\|\chi\|_S\|\eta\|_S \quad \forall \chi, \eta \in S_h.$$  

Therefore, (55) implies a uniform inf-sup condition for $A_h$ on $S_h$ and hence existence and uniqueness of discrete solution, i.e., to (9). Moreover, standard arguments and Proposition 11 show that

$$\|\xi_h - \xi_h^n\|_S \leq Ch(||f||_{1,\infty,\Omega^{-}} + \|\xi_h\|_S).$$  

Then, stability of the Galerkin method proves the first estimate and Theorem 5 proves the second one.
REFERENCES